Hugenholtz-Van Hove Theorem for Multi-Component
Fermi Systems with Multi-body Forces

*R. C. Nayak and S. Pattnaik*
Dept. of Physics, Berhampur University, Berhampur-760007, INDIA

Introduction

The Hugenholtz-Van Hove (HVH) theorem [1] deals with the single-particle properties of an interacting infinite Fermi system at absolute zero of temperature relating three fundamental physical quantities namely, the average energy per fermion $E/A$, the pressure of the system and the Fermi energy $\epsilon_f$ as

$$\frac{E}{A} + \rho \frac{\partial (E/A)}{\partial \rho} |_{\Omega} = \left( \frac{\partial E}{\partial A} \right)_\Omega,$$

(1)

where $\rho$ and $\Omega$ are respectively the number density and volume of the system. Hugenholtz and Van Hove also showed

$$\left( \frac{\partial E}{\partial A} \right)_\Omega = \epsilon_f,$$

leading to the theorem

$$\frac{E}{A} + \rho \frac{\partial (E/A)}{\partial \rho} |_{\Omega} = \epsilon_f.$$  

(2)

The second term being the pressure of the system would vanish for a saturating system at ground state i.e. at equilibrium resulting in the Eq. $E/A = \epsilon_f$. This is a rare theorem in many-body Physics, which has been rigorously shown[1] to be true by its original authors Hugenholtz and Van Hove by taking all orders of perturbation in the framework of time-independent perturbation theory[1–5]. However it should be mentioned here that prior to this rigorous proof Bethe[6] had also visualized the theorem under HF approximation. It is valid for any interacting infinite Fermi system and thereby applicable to liquid $^3$He and in particular to nuclear matter. With its help Hugenholtz and Van Hove could find[1] internal inconsistencies in the early nuclear matter calculations of Brueckner [7]. Apart from its utility otherwise, Hugenholtz and Van Hove while proving the theorem have clearly brought out the physical meaning associated with the single particle states of an interacting many-fermion system.

The theorem has been recently extended [8] to asymmetric nuclear matter, which was then used for constructing a successful mass model well-known in the literature as the infinite nuclear matter (INM) model of atomic nuclei[9–12]. However the question of its validity in the presence of multi-body interaction terms remains unanswered. Similarly its extension to multi-component Fermi systems would be extremely useful.

HVH Theorem with Multi-body Forces

For this we follow Bethe[6] in adopting HF approximation. Taking the effective interaction for a many-body system to include all possible multi-body interaction terms $G_2, G_3, G_4, ..., G_n$, etc., the single-particle energy $\epsilon_i$ and the total energy $E$ of the system under HF approximations are

$$\epsilon_i = \langle i | -\frac{\hbar^2}{2m} \nabla^2 | i \rangle + \sum_j < ij | G_2 | ij > + \sum_{ij} < ij \cdots n | G_n | ij \cdots n >$$

(3)

and

$$E = \sum_i < i | -\frac{\hbar^2}{2m} \nabla^2 | i \rangle + \frac{1}{2} \sum_{ij} < ij | G_2 | ij > + \sum_{ij} < ij \cdots n | G_n | ij \cdots n >.$$  

(4)

*Electronic address: rcnayak00@yahoo.com

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where $|i>$, $|j>$ etc. represent the occupied single particle states and $< i..n | G_n | i..n >$ denote the antisymmetric many-body matrix elements with the multi-body interaction $G_n$. For an infinite Fermi system, the above Eqs. can be further simplified as

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m} + \frac{\Omega}{(2\pi)^3} \int_0^{k_f} <kk'|G_2|kk'> d^3k' + \ldots + \frac{1}{(n-1)!} \left[ \frac{\Omega}{(2\pi)^3} \right]^{n-1} \int_0^{k_f} \ldots \int_0^{k_f} <kk_{n-1}k_n|G_n|kk_{n-1}k_n > d^3k_{n-1}d^3k_n$$

and

$$E = \frac{\Omega}{(2\pi)^3} \int_0^{k_f} \frac{\hbar^2 k^2}{2m} d^3k + \frac{1}{2} \left[ \frac{\Omega}{(2\pi)^3} \right]^2 \int_0^{k_f} \ldots \int_0^{k_f} d^3k_1d^3k_2 \ldots d^3k_n <k_1..k_n|G_n|k_1..k_n>$$

where $\hbar^2 k^2/2m$ is the kinetic energy and $\int_0^{k_f} d^3k$ is the volume integral. The total energy $E/A$ being a function of $k_f$, its derivative with respect to $k_f$ at constant $\Omega$ can be easily obtained for all the multi-body interaction terms thereby leading to the HVH theorem (Eq. 2).

**Extension to Multi-Component Fermi Systems**

Consider the system to consist of $n$ types of fermions, the number of each type being $N_i, i = 1, 2, n$, such that the total number $N$ is equal to $N_1 + N_2 + \ldots + N_n$. Defining the fractional composition $f_i$ by $f_i = N_i/N, i = 1, 2, n$, then the system will have $n$ Fermi energies given by

$$\epsilon_i^f = \left( \frac{\partial E}{\partial N_i} \right)_{\Omega,N_1..N_{i-1},N_{i+1}..N_n}$$

As the total energy $E$ being a function of $N_1, N_2, \ldots, N_n$ or alternatively $N, f_1, f_2, \ldots f_n$, it follows that

$$\left( \frac{\partial E}{\partial N} \right)_{\Omega} = \left( \frac{\partial E}{\partial N_1} \right)_{\Omega,N_2,N_3..} \left( \frac{\partial N_1}{\partial N} \right) f_1 + \left( \frac{\partial E}{\partial N_2} \right)_{\Omega,N_1,N_3..} \left( \frac{\partial N_2}{\partial N} \right) f_2 + \ldots = \sum_{i=1}^{n} \epsilon_i^f f_i$$

Using Eq. (1), we arrive at the most generalized form of HVH theorem as

$$\frac{E}{A} + \rho \frac{\partial (E/A)}{\partial \rho} = \sum_{i=1}^{n} \epsilon_i^f f_i$$

which for ground state reduces to

$$\frac{E}{A} = \sum_{i=1}^{n} \epsilon_i^f f_i$$

It is needless to mention the utility of such a generalized HVH theorem that can be applied to any multi-component Fermi system.

**References**